THE FUNCTION $W = L(\tau)$ DEFINED AS THE INVERSE OF THE FUNCTION $\tau = W - \log W$.

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In the preceding paper W has been defined as a function of τ by the relation $W - \log W = \tau$, and this has been designated by the notation $W = L(\tau)$. We propose here to study the nature of this function.

This situation is analogous to the one defined by Kepler's Equation, $M = E - e \sin E$, in the problem of two bodies, where E is desired as a function of M.

From the point of view of the theory of functions $W = L(\tau)$ is an infinitely many-valued function of τ, and likewise τ is an infinitely many-valued function of W. The branch-points of $L(\tau)$ are determined by the equations

$$(1) \qquad \begin{aligned} W - \log W &= \tau, \\ 1 - \frac{1}{W} &= 0, \end{aligned}$$

from which we find $W = 1, \tau = 1 - 2n\pi i$; and since the second derivative does not vanish for $W = 1$ only two branches interchange at the branch points. For the purpose of examining the complex values of the function we will take

$$W = \rho e^{i\phi}, \qquad \tau = \alpha + i\beta,$$

which substituted in the first equation of (1) gives the two equations

$$(2) \qquad \begin{aligned} \rho \cos \varphi - \log \rho &= \alpha, \\ \rho \sin \varphi - \varphi &= \beta. \end{aligned}$$

Let us suppose now that W moves around the origin in a circle in the W-plane; then the value of ρ is constant and equations (2) show that τ describes a trochoid in the τ-plane, the radius of the rolling circle being equal to unity, the τ-point being at a distance ρ from its center, and the axis of rolling being the line $\alpha = 1 - \log \rho$. When ρ is greater than unity the trochoid has loops, each of which encloses a branch-point.

In order to have a one-to-one correspondence between the values of W and τ we will construct a Riemann surface for each of the variables. The equation $W - \log W = \tau$ is already solved for τ as a function of W. It is clear that τ will be a uniform function of W upon a W-surface of

infinitely many sheets which wind about the origin, a single sheet being included in the range $2n\pi \leqq \varphi < 2(n+1)\pi$.

Let us consider the values of τ for values of W lying in the principal sheet, $\rho \cos \varphi - \log \rho = \alpha_1$, $\rho \sin \varphi - \varphi = \beta_1$. Suppose W describes a circle for which ρ is small and $0 \leqq \varphi < 2\pi$. The corresponding path of τ is very nearly a straight line parallel to the β-axis connecting the points $(\rho - \log \rho, 0)$ and $(\rho - \log \rho, -2\pi)$ but this last end point excluded. For larger values of ρ, this line, which in reality is a trochoid described by a point near the center of the rolling circle, rapidly approaches the β-axis and increases slightly in curvature. For ρ equal to unity the trochoid becomes a cycloid connecting the two branch-points $(1, 0)$ and $(1, -2\pi)$. Consequently the interior of a circle of unit radius in the W-plane maps into a strip in the τ-plane of width 2π bounded by and including the real axis from the point $(1, 0)$ to $(+\infty, 0)$ and bounded by and excluding the straight line $\beta = 2\pi$, from the point $(1, -2\pi)$ to $(+\infty, -2\pi)$ and the cycloid connecting the two points $(1, 0)$ and $(1, -2\pi)$.

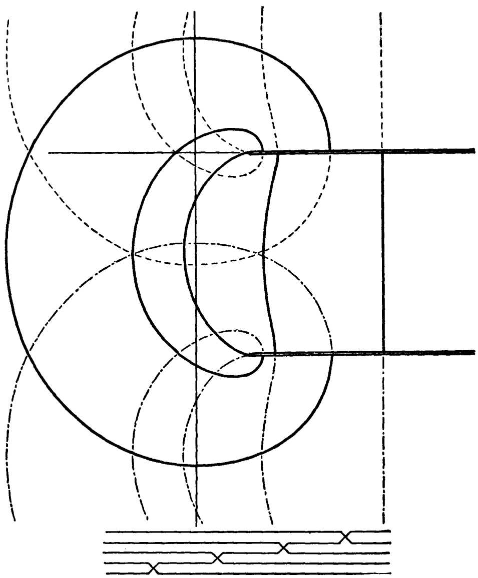

FIG. 1.

In order to avoid a double line in the τ-plane we will take for $\rho > 1$ the range $0 < \varphi \leqq 2\pi$. There a circle of radius slightly greater than unity maps into a trochoid of curvature greater than the cycloid connecting the points $(\rho - \log \rho, 0)$ and $(\rho - \log \rho, -2\pi)$, the first point excluded. As ρ increases the trochoid expands, the initial and end points receding

from the β axis. For very large values of ρ the trochoid becomes very nearly the arc of a circle connecting the points $(\rho - \log \rho, 0)$ and $(\rho - \log \rho, -2\pi)$, the radius of the circle being equal to ρ. Clearly the exterior of the unit circle in the principal W-sheet maps into the region bounded by and excluding the real axis from $(1, 0)$ to $(+\infty, 0)$ and the above-mentioned cycloid and bounded by and including the straight line $\beta = -2\pi$ between the points $(1, -2\pi)$ and $(+\infty, -2\pi)$, lying on the outside of these boundaries. Thus the principal sheet of the W-surface maps into an entire sheet in the τ-surface. The two straight lines described in the above boundaries are lines of discontinuity, that is, as τ crosses these lines the corresponding values of W change abruptly. Consequently we will suppose the τ sheet to be cut along these lines.

Consider now the values of τ corresponding to the values of W in the second sheet, for which $2\pi \leqq \varphi < 4\pi$. They are defined by the equations

$$\rho \cos \varphi - \log \rho = \alpha_2,$$

$$\rho \sin \varphi - \varphi = \beta_2 + 2\pi.$$

Thus we have for the same values of ρ and φ, $\alpha_1 = \alpha_2$, $\beta_1 - 2\pi = \beta_2$, and the second τ-sheet differs from the first only in that the β coördinates are diminished by 2π. Let similar cuts be made in the second τ-sheet from the points $(1, -2\pi)$ and $(1, -4\pi)$ parallel to the real axis to $+\infty$. If the second τ-sheet be superposed upon the first the two sheets can be joined along the cuts proceeding from the points $(1, -2\pi)$, the lower edge in the first sheet being joined to the upper edge in the second sheet and the upper edge of first sheet to the lower edge of the second. If now τ crosses this line moving from the first sheet to the second, W will move continuously from the first W-sheet to the second—and conversely—the two sheets are connected by the single branch-line $(1, -2\pi)$ to $(+\infty, -2\pi)$.

In the same way a third τ-sheet can be constructed corresponding to the third W-sheet which can be superposed upon the second and connected with it along the line proceeding from the branch-point $(1, -4\pi)$; and so on, indefinitely. The τ-surface has infinitely many sheets, each sheet has two branch-points, $(1, -2n\pi)$ and $(1, -2(n+1)\pi)$, the first connecting it with the sheet below and the second connecting it with the sheet above. If W describes a circle about the origin passing from sheet to sheet so that φ always increases, the trochoid described by τ will pass upwards through the sheets of the τ-surface. There is a one-to-one correspondence between the sheets of the two surfaces, and a one-to-one correspondence between the values in the two corresponding sheets.

Real values of the functions $L(\tau)$ and its derivative $L'(\tau)$ corresponding

to real values of the independent variable τ are given in the accompanying tables for that branch of the function for which the derivative is positive.

VALUES OF $L(\tau)$.

τ	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1.......	1.000	1.516	1.772	1.986	2.179	2.357	2.527	2.690	2.846	2.998
2.......	3.146	3.291	3.433	3.573	3.711	3.847	3.981	4.114	4.246	4.376
3.......	4.505	4.633	4.761	4.886	5.011	5.136	5.261	5.384	5.506	5.628
4.......	5.749	5.870	5.990	6.110	6.229	6.348	6.467	6.585	6.702	6.820
5.......	6.937	7.053	7.170	7.286	7.401	7.517	7.632	7.747	7.862	7.976
6.......	8.091	8.205	8.318	8.432	8.545	8.658	8.771	8.884	8.997	9.109
7.......	9.221	9.334	9.446	9.557	9.669	9.780	9.892	10.003	10.114	10.225
8.......	10.336	10.447	10.557	10.667	10.777	10.887	10.997	11.107	11.217	11.327
9.......	11.437	11.547	11.657	11.766	11.875	11.984	12.093	12.202	12.311	12.420
10.......	12.529	....	....	....	....	....	....	....	....	....

$L(100) = 104.651, \; L(1000) = 1006.915$

VALUES OF $L'(\tau)$.

	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
1.......	∞	2.9376	2.2950	2.0140	1.8482	1.7369	1.6549	1.5917	1.5417	1.5005
2.......	1.4660	1.4365	1.4110	1.3887	1.3689	1.3512	1.3355	1.3211	1.3081	1.2962
3.......	1.2853	1.2753	1.2659	1.2573	1.2493	1.2418	1.2347	1.2281	1.2219	1.2160
4.......	1.2106	1.2053	1.2004	1.1957	1.1912	1.1870	1.1829	1.1790	1.1754	1.1718
5.......	1.1684	1.1652	1.1621	1.1591	1.1562	1.1535	1.1508	1.1482	1.1457	1.1434
6.......	1.1410	1.1388	1.1366	1.1346	1.1325	1.1306	1.1287	1.1268	1.1250	1.1233
7.......	1.1216	1.1200	1.1184	1.1169	1.1154	1.1139	1.1125	1.1111	1.1097	1.1084
8.......	1.1071	1.1059	1.1046	1.1034	1.1023	1.1011	1.1000	1.0989	1.0978	1.0968
9.......	1.0958	1.0948	1.0938	1.0929	1.0919	1.0910	1.0902	1.0893	1.0884	1.0876
10.......	1.0867	....	....	....	....	....	....	....	....	....

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